J. Fluid Mech. (1983), *vol.* 132, *pp.* 417–430
Printed in Great Britain

A resonant test-field model of gravity waves

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(Received 10 September 1981 and in revised form 22 February 1983)

In this paper we propose an 'irreversible' resonant test-field (RTF) model to describe the statistical fluctuations of gravity waves on deep water driven by a turbulent wind field. The non-resonant interactions in the gravity-wave Hamiltonian are replaced by a Markov process in the equation of motion for the resonantly interacting gravity waves, i.e. Hamilton's equations are replaced by a Langevin equation for the RTF waves. The RTF models the irreversible energy-transfer process by a Fokker–Planck equation for the phase-space probability density, the *exact* steady-state solution of which is determined to be non-Gaussian. An H-theorem for the RTF predicts the monotonic approach to the asymptotic steady state near which the transport properties of the field are studied. The steady-state energy-spectral density is calculated (in some approximation) to be k^{-4}.

1. Introduction

In this paper we propose an 'irreversible' resonant test field (RTF) model to describe the statistical fluctuations of gravity waves on deep water. This model implements a Markov approximation for the statistics of the random gravity wave field and includes the energy flux from the turbulent wind field, the resonant interaction among the gravity waves and the non-resonant wave–wave interactions acting as a 'heat bath.' The RTF model is the logical successor to the linear Gaussian wave fields used previously to describe the evolution of waves on deep water using weak-interaction theory (see e.g. Hasselmann 1962, 1963, 1967; Valenzuela & Laing 1972; Weber & Barrick 1977*a*, *b*). In these and other studies, the statistics of the nonlinear water waves were assumed to be a known rather than a derived property of the field. In the present analysis, the wave dynamics are used to determine the statistics of the gravity wave field, and consequently their statistics cannot be known *a priori*. Hasselmann (1967) has argued that a random linear wave field can be regarded as Gaussian and that weak nonlinear interactions will not counteract the linear tendency for the wave field to be Gaussian. This contention contradicts the generalization of a theorem due to Doob and proven by Lax (1966*a*) which states: *a random process that is Gaussian and Markovian must be a linear, Fokker–Planck process*. Thus one must relinquish either Gaussianity or nonlinearity in order to describe water waves as a Markov random process.

In the weak-interaction theory of water waves, the observables at the ocean surface are expressed in Fourier series. An individual linear water wave is labelled by a wave vector $\boldsymbol{k}$, a frequency ω_k and has a mode amplitude $C_{\boldsymbol{k}}$. In a linear wave field these mode amplitudes are constant in time. In a weakly nonlinear wave field the wave–wave interactions couple these modes together, thereby providing a mechanism for energy and momentum exchange and the mode amplitudes become time dependent.

Zakharov (1968) was the first to show that an isolated field of gravity waves is a conservative Hamiltonian system, so that the evolution of the mode amplitudes are given by Hamilton's equations of motion. The Hamiltonian for the gravity-wave system is a series in which the nonlinear terms appear as products of the mode amplitudes. These interactions induce a variation in both the amplitudes and phases of the linear waves which is much slower than the harmonic variation of the linearized system.

The interactions in an arbitrary Hamiltonian system can be separated into those that are resonant and those that are non-resonant (for a complete discussion see e.g. Moser 1973). It has long been assumed that the resonant interactions dominate the evolution of water waves (see e.g. Phillips 1960; Longuet-Higgins 1962; Benney 1962), so that non-resonant interactions were either treated in perturbation theory (see e.g. Watson & West 1975), or ignored entirely. The non-resonant interactions were thought to be of secondary importance because these interactions result in many changes of sign of a wave–wave coupling in the characteristic time interval for the mode amplitude to change sensibly. We argue in §2 that this rapid oscillation can be replaced by a statistical fluctuation driving the $\boldsymbol{k}$-wave. The model equation of evolution for the $\boldsymbol{k}$-wave therefore consists of the resonant interactions and an additive fluctuating force mimicking the dynamics of the non-resonant interactions. This partitioning of effects suggests a Langevin description of the surface waves which is similar in spirit to the Langevin description for internal-wave dynamics used by Pomphrey, Meiss & Watson (1980). However, because a given wave can in general participate in both a resonant interaction with some waves and a non-resonant interation with others, we introduce a two-field model. The first field consists of test waves that can interact resonantly among themselves. This has been one of the most successful models of the nonlinear interactions used in many-body systems in the past. The second field consists of waves that interact non-resonantly among themselves and which are at most linearly coupled to the resonant test-wave field. West (1982*a*) has shown that this two-field model can be represented by a nonlinear Langevin equation for the test-wave modes in the Markov limit.

The Langevin equation description of the gravity wave field is a *nonlinear stochastic rate equation* whose analytic solution for an arbitrary initial configuration of waves is not known. The fluctuations in the Langevin equation arise from two sources in the present model: (1) the non-resonant interactions in the wave field, and (2) the turbulent air flow in the region of the sea surface. The turbulent air flow would be sufficient to generate fluctuations in the dynamic response of the sea surface; however, there is no known air–sea coupling mechanism that will dissipate the surface-wave energy. The dissipation of surface-wave energy is necessary to provide the observed asymptotic statistical steady state for the gravity-wave field. In the present model the average of the fluctuating flux in the Langevin description of the test-wave field provides a dissipative current which eventually balances both the energy flux from the wind field and that transferred from other regions of the spectrum by the wave–wave interactions. The development of this steady state is discussed in §3.

In an evolving field of water waves, the statistics evolve coincidentally with the amplitudes and phases of the individual modes. It is the evolution of the statistics which determines how the physical observables are transported in space and time. We determine how the statistics evolve *without* constructing a hierarchy of cumulant transport equations. A hierarchy method based on singular-perturbation theory has been developed by Benney & Saffman (1966), Benney & Newell (1967) and Newell

(1968). We avoid this approach by replacing the nonlinear stochastic rate equations by the phase-space equation of evolution for the two-time-point conditional probability density for the gravity-wave field. The two-point probability density function describes an ensemble of possible paths of evolution realizable by the RTF in phase space and, since the wave field is assumed to be Markovian, all the statistical information is contained in this two-point probability density. This assumption is the next logical step beyond the statistical hypothesis of Gaussian mode amplitudes made by other investigators since this latter assumption only concerns the single-point probability density.

The phase-space equation governing the evolution of the probability density is the Fokker–Planck equation. Although we cannot solve this equation for the full time-dependent probability density, an *exact* steady-state solution is obtained. The steady-state probability density is found to be non-Gaussian, and estimates of the asymptotic energy spectral density based on this distribution are found to agree with the laboratory and field data, as compiled, for example, by Kitaigorodskii (1970) and Mitsuyasu (1975). For gravity waves the spectrum is determined to be proportional to k^{-4}. This is the first *dynamic* model which yields this spectrum.

2. The equations of motion

In this section we describe the motion of the surface of an idealized ocean, i.e. we treat the ocean as a large basin of water and use the equations of motion for an inviscid, irrotational fluid described by a velocity potential $\phi(\boldsymbol{x}, z, t)$ and surface deflection $z = \zeta(\boldsymbol{x}, t)$. The basin is assumed to be large in lateral extent with horizontal coordinates $\boldsymbol{x}$ and very much deeper than the longest characteristic scale of the surface motion. These two assumptions allow us to ignore the effect of the fixed boundaries of our basin on the motion of the surface and indeed to separate the motion of the surface completely from the interior fluid motion. Zakharov (1968) has demonstrated that the surface deflection $\zeta(\boldsymbol{x}, t)$ and the velocity potential $\phi_s(\boldsymbol{x}, t)$ at the free surface of the fluid constitute a set of canonical field variables and that the equations of motion of this surface follow from Hamilton's principle of least action. In terms of these canonical variables the Hamiltonian for the gravity-wave field can be written

$$H_g = \sum_{j=2}^{\infty} \epsilon^j H_j, \tag{2.1}$$

where ϵ is an ordering parameter and the subscript on H_j indicates the order of the product of the canonical field variables in this piece of the Hamiltonian (for details see e.g. Monin, Kamenkovich & Kort 1974; Broer 1974; Miles 1977; Milder 1977).

As in West (1982*b*; hereinafter referred to as Part 1) we find it convenient to express the equations of motion in terms of the linear eigenmode amplitudes of the fluid surface. For the present isolated system these are just the Fourier mode amplitudes and constitute a canonical transformation from the field variables $[\zeta(\boldsymbol{x}, t), \phi_s(\boldsymbol{x}, t)]$ to the set of canonical mode amplitudes $\{c_{\boldsymbol{k}}(t)\}$. The equations of motion for the gravity-wave field are given by

$$\dot{C}_{\boldsymbol{k}} = -\mathrm{i}\frac{\partial H_g}{\partial C^*_{\boldsymbol{k}}}, \tag{2.2}$$

and using the series expansion for the Hamiltonian (2.1) we obtain

$$\dot{C}_{\boldsymbol{k}}(t) + \mathrm{i}\omega_k C_{\boldsymbol{k}}(t) = T_{\boldsymbol{k}}(C, t). \tag{2.3}$$

The function $T_{\boldsymbol{k}}(C)$ is the sum of the quadratic and cubic wave–wave interactions as described in Part 1. However, since the present field is assumed to be isolated, the coupling coefficients in the interaction terms are somewhat different from those developed in Part 1 (see e.g. Watson & West 1975; West 1981*a*).

Observations of the surface of the open ocean are not consistent with the instantaneous, deterministic properties of the surface predicted by (2.3). The data on the vertical displacement of the fluid and the surface velocity indicate that measurements of the surface properties at one space–time point are not sufficient to predict these same properties at a nearby space–time point. The consensus of opinion is that the water waves are most consistently represented by a *stochastic nonlinear wave field* rather than one that is deterministic (see e.g. Kinsman 1965; Kitaigorodskii 1970; Phillips 1977; West 1981*a*). This difficulty in using (2.3) has usually been circumvented by *assuming* that the mode amplitudes $C_k(t)$ are stochastic quantities. Hasselmann (1962, 1963) assumed the mode amplitudes to be Gaussian random variables providing a spatially homogeneous spectrum of gravity waves. Watson & West (1975), Willebrand (1975) and Alber (1978) made similar statistical assumptions for an inhomogeneous spectrum of gravity waves. The fluctuations in the turbulent wind field driving the water waves of course account in part for the fluctuations in the surface properties (see e.g. Phillips 1977; West & Seshadri 1981). However, here we are interested in the fluctuations attributable to the nonlinear interactions within the wave field itself in addition to those generated by external sources.

Here we propose a model in which the non-resonant interactions among the gravity waves act as a source of fluctuations in the evolution of the wave field. Such a model implies that there are at least two characteristic timescales in the equation of motion; the timescale for the fluctuations and the much longer timescale for the average or macroscopic development of the wave field. The Hamiltonian (2.1) has two distinct perturbation contributions; one from resonant and the other from non-resonant interactions. The timescale for energy transfer to a $\boldsymbol{k}$-wave through a resonant interaction is roughly of the order of $\tau_k^r \approx 1/\omega_k \epsilon^2$. The timescale for the periodic transfer of energy in and out of the $\boldsymbol{k}$-wave through a non-resonant interaction is of the order $\tau_k^s \sim 1/\omega_k \epsilon$. Thus $\tau_k^s \approx \epsilon \tau_k^r$, where ϵ is a measure of the surface slope, and in the deep ocean $\epsilon \leqslant 0.05$, indicating that the variations in the mode amplitudes due to the non-resonant interactions are an order of magnitude more rapid than those due to resonant terms.

If one were to attempt to integrate the equations of motion (2.3) directly, one would encounter the same problems of resolution usually associated with turbulent fluid flow. The separation in space- and timescales, in addition to their theoretical difficulties, make the calculations extremely expensive. We therefore elect to *model* the dynamics of the gravity-wave field by generalizing the concept of a test wave used successfully by Pomphrey *et al.* (1980) to describe the relaxation of a test wave in an equilibrated field of internal waves. In that model the test wave can only interact singly with the waves in the ambient field.† Here, to mimic the effect of the timescale separation, we assume that the average properties of the surface-wave field can be described by n resonantly interacting test waves linearly coupled to a field on non-resonantly interacting modes. The rapid internal fluctuations in (2.3) are replaced by an external fluctuating flux. We refer to this partitioning of effects as a *resonant test-field model* (RTF) and we adopt the pragmatic point of view that its justification will be determined by its utility.

† See the conference proceedings edited by West (1981*b*) for a critique of this model used to describe the dynamics of internal waves.

The RTF model represents a broadband spectrum of surface waves by two distinct wave fields, i.e. the set $\{C_k(t)\}$ is separated into the waves $\{A_k(t)\}$ and $\{B_\nu(t)\}$. The ambient wave field $\{B_\nu(t)\}$ is an equilibrated spectrum of waves that interact non-resonantly among themselves and couple linearly to a second field of waves $\{A_k(t)\}$. This second field has a discrete spectrum of waves that interact resonantly with each other and experience the ambient waves as a source of additive fluctuations. The mean value of these fluctuations determines the average coupling between the two wave fields.

The total Hamiltonian has the form

$$H_g = H_R(\boldsymbol{A}) + H_a(\boldsymbol{B}) + H_{aR}(\boldsymbol{A}, \boldsymbol{B}). \tag{2.4}$$

The quantity $H_R(\boldsymbol{A})$ consists of the resonant test waves and can be written

$$H_R = \sum_k \omega_k A_k A_k^* + V_R, \tag{2.5}$$

where V_R is the nonlinear resonant-interaction potential. The Hamiltonian for the non-interacting ambient waves and the coupling between these waves and the RTF waves is

$$H_a + H_{aR} = \sum_\nu \omega_\nu [B_\nu + \mathrm{i} G_\nu(\boldsymbol{A}, \boldsymbol{A}^*)][B_\nu^* - \mathrm{i} G_\nu^*(\boldsymbol{A}, \boldsymbol{A}^*)], \tag{2.6}$$

where $G_\nu(\boldsymbol{A}, \boldsymbol{A}^*)$ is a function describing the modulation of the ambient wave field by the RTF waves.

Hamiltons equations of motion for this partitioned system replaces (2.2) with

$$\dot{B}_\nu = -\mathrm{i} \frac{\partial H_g}{\partial B_\nu^*} \tag{2.7}$$

for the ambient waves, and

$$\dot{A}_k = -\mathrm{i} \frac{\partial H_g}{\partial A_k^*} \tag{2.8}$$

for the RTF waves. Using the Hamiltonian (2.4)–(2.6) the equations of motion for the ambient waves are

$$\dot{B}_\nu + \mathrm{i}\omega_\nu B_\nu = \omega_\nu G_\nu(\boldsymbol{A}, \boldsymbol{A}^*), \tag{2.9}$$

and those for the RTF waves are

$$\dot{A}_k + \mathrm{i}\omega_k A_k = -\mathrm{i} \frac{\partial V_R}{\partial A_k^*} + \sum_\nu \omega_\nu \left\{ (B_\nu + \mathrm{i} G_\nu) \frac{\partial B_\nu^*}{\partial A_k^*} - (B_\nu^* - \mathrm{i} G_\nu^* \frac{\partial B_\nu}{\partial A_k^*} \right\}. \tag{2.10}$$

Taken together, (2.9) and (2.10) constitute a feedback system between the two wave fields. West (1982*a*) has constructed a description of the evolution of the resonant test field by solving (2.9) and using this exact solution to eliminate the dependence of (2.10) on the ambient field variables.

We choose for our model a linear modulation of the ambient waves by writing

$$G_\nu \equiv \sum_p \Gamma_{\nu p} A_p, \tag{2.11}$$

where $\Gamma_{\nu p}$ is a complex coupling coefficient. With this choice of G_ν, West (1982*a*) has shown that for an ensemble of initial states of the ambient waves in equilibrium configurations characterized by the distribution

$$P(\boldsymbol{B}(0)|\boldsymbol{A}(0)) \sim \exp\left\{ -\sum_\nu (H_a + H_{aR})(\nu)/\Lambda_\nu \right\} \tag{2.12}$$

(with the RTF variables held fixed at time $t=0$) (2.10) reduces in the Markov limit to

$$\dot{A}_k(t)+(\lambda_k+i\omega_k)A_k(t) = -\left(\frac{\lambda_k+i\omega_k}{\omega_k}\right)\frac{\partial V_R}{\partial A_k^*}+f_k^N(t). \tag{2.13}$$

Equation (2.13) is the RTF equation of motion in the Markov approximation with the fluctuating flux $f_k^N(t)$ defined by

$$f_k^N(t) \equiv \sum_v \Gamma_{vk}^* \omega_v [B_v(0)+iG_v(A,0)]\, e^{-i\omega_v t}. \tag{2.14}$$

Because the distribution of initial states (2.12) is a multivariate Gaussian in the quantity $[B_v(0)+iG_v(A,0)]$ (cf. (2.6)), the fluctuations (2.14) are zero-centred and in the present approximation are delta-correlated in time.

The coupling of the RTF to the ambient wave field has modified the original Hamiltonian equation in three ways: (1) there is a zero-centred Gaussian fluctuating flux $f_k^N(t)$ driving the wave field; (2) the Hamiltonian character of the system is lost owing to the dissipative flux λ_k of action (energy) to the ambient waves; and (3) there is a modification of the nonlinear interactions due to a back-reaction of the ambient waves to the nonlinear interactions among the test waves.

It is now a simple matter to include the effect of the air flow generating a field of water waves by modifying (2.13) to incorporate a linear coupling between the wind and the ocean surface. Introducing the Miles (1957) air–sea coupling parameter μ_k, which models the average in-phase coupling of the air flow to the fluid surface, and Phillips's (1957) incoherent fluctuations in the pressure field (see e.g. Part 1),

$$f_k^W(t) = i\frac{p_k(t)}{\rho_W V_k}, \tag{2.15}$$

where $p_k(t)$ is the Fourier transform of the fluctuating pressure field at the sea surface, we rewrite (2.14) as

$$\dot{A}_k(t)+[\lambda(k)+i\omega_k]A_k(t) = f_k(t)+T_k(A,t). \tag{2.16}$$

In (2.16) we have defined

$$\lambda(k) \equiv \lambda_k - \mu_k\omega_k, \tag{2.17}$$

$$f_k(t) \equiv f_k^W(t)+f_k^N(t), \tag{2.18}$$

$$T_k(A) \equiv -\left(\frac{\lambda(k)+i\omega_k}{\omega_k}\right)\frac{\partial V_R}{\partial A_k^*}. \tag{2.19}$$

Equation (2.16) is the nonlinear Langevin equation modelling the evolution of the gravity-wave field used in the remainder of this work.

To determine the properties of the solution to (2.16), the full statistics of the stochastic driver $f_k(t)$ must be given. The mean value of $f_k(t)$ is zero,

$$\overline{f_k(t)} = 0, \tag{2.20a}$$

since that of f^N is zero by construction and that of f^W is zero by hypothesis. The second moments of $f_k(t)$ are given by

$$\overline{f_k^W(t)f_{k'}^{W*}(t-\tau)} = 2D_k^W\,\phi_k^W(\tau)\,\delta_{k-k'}, \quad \overline{f_k^W(t)f_k^W(t-\tau)} \approx 0, \tag{2.20b}$$

$$\overline{f_k^N(t)f_{k'}^{N*}(t-\tau)} = 2D_k^N\,\phi_k^N(\tau)\,\delta_{k-k'}, \quad \overline{f_k^N(t)f_k^N(t-\tau)} \approx 0, \tag{2.20c}$$

$$\overline{f_k^W(t)f_{k'}^{N*}(t-\tau)} = 0, \tag{2.20d}$$

where D_k^W is the spectrum of fluctuations in the air flow and $\phi_k^W(\tau)$ is the correlation of these fluctuations over a time interval τ. A similar interpretation applies to D_k^N and $\phi_k^N(\tau)$. Also the fluctuating flux f_k^N in the wave field is assumed to be statistically independent of the fluctuations f_k^w in the air flow, both of which are statistically homogeneous. For tractability we assume that the higher-order cumulants (indicated by a double overbar) of $f_k(t)$ are zero,

$$\overline{\overline{f_k(t_1)\dots f_k(t_n)}} = 0 \quad (n > 2), \tag{2.20e}$$

so that $f_k(t)$ is a Gaussian random process, i.e. it is the sum of two such processes f_k^w and f_k^N. Considerable simplification results from treating the ambient wave interactions as delta-correlated in time, independent of spatial scale, i.e.

$$\phi_k^N(t) \approx \delta(t). \tag{2.21}$$

The assumptions on the statistical properties of the fluctuating function $f_k(t)$ given by (2.20) and (2.21) specify a Markov process. The response of the wave field to these fluctuations is determined by (2.16), which constitutes a set of *nonlinear stochastic rate equations* for the resonant test-field model of the gravity-wave mode amplitudes. The solution to such a set of stochastic equations is obtained either by direct integration (prohibitively difficult for water-wave systems with many waves) or from the two-point probability density function describing an ensemble of possible paths of evolution realizable by the test system in phase space. We assume that the response of the test field to the fluctuations can be satisfactorily described by a Markov process, i.e. that all the statistical information about the field is contained in the two-point probability density. Also we do not assume the form for the two-point probability density, but rather determine its equation of evolution and deduce its form from the properties of the exact solution to this equation.

3. The Fokker–Planck equation

The equation of evolution (2.16) describes the development of the set of dynamic variables $A(t) \equiv \{A_k(t)\}$ for a particular realization of the set of fluctuations $\{f_k(t)\}$. One can define a phase space $\Gamma(\boldsymbol{a})$ for the dynamical vector $A(t)$ by the values $\boldsymbol{a}$ that the vector can assume. For each realization of the additive fluctuations $\{f_k(t)\}$ there corresponds a unique trajectory in this phase space which describes the evolution of the water wave field. A large number of realizations of $f(t)$ defines a corresponding ensemble of trajectories in the phase space. This ensemble of test-wave fields can be described by a two-time-point probability density which determines all the information that can be experimentally known about the wave field when the process is Markov. In this section we construct the equation of evolution for $P(\boldsymbol{a}, t|\boldsymbol{a}_0)$, where $P(\boldsymbol{a}, t|\boldsymbol{a}_0)\,\mathrm{d}\Gamma(\boldsymbol{a})$ is the probability that $A(t)$ has a value in the interval $(\boldsymbol{a}, \boldsymbol{a}+\mathrm{d}\boldsymbol{a})$ at time t given an initial value $\boldsymbol{a}_0$, and $\mathrm{d}\Gamma(\boldsymbol{a})$ is a differential volume element of phase space.

For zero-centred Gaussian fluctuations, delta-correlated in time, the phase-space equation of evolution for the two-point probability density can be shown by standard arguments (see e.g. Lax 1966*b*, van Kampen 1976, Lindenberg *et al.* 1983) to be the Fokker–Planck equation. In terms of the complex mode amplitudes the Fokker–Planck equation corresponding to (2.16) is

$$\frac{\partial}{\partial t}P(\boldsymbol{a}, t|\boldsymbol{a}_0) = \sum_k \frac{\partial}{\partial a_k}\left[\left(\frac{\lambda(\boldsymbol{k})+\mathrm{i}\omega_k}{\omega_k}\right)\frac{\partial H_\mathrm{R}}{\partial a_k^*}P(\boldsymbol{a}, t|\boldsymbol{a}_0)\right] + \text{c.c.} + \sum_k 2D_k \frac{\partial^2}{\partial a_k \partial a_k^*}P(\boldsymbol{a}, t|\boldsymbol{a}_0), \tag{3.1}$$

where H_R is the RTF Hamiltonian given by (2.5), the diffusion coefficient is

$$D_k \equiv D_k^N + \frac{\tau_c(\boldsymbol{k})\, D_k^W}{\rho_0^2\, V_k^2}, \tag{3.2}$$

and $\tau_c(\boldsymbol{k})$ is the correlation time of a fluctuation of wavelength $2\pi/k$ in the pressure field, i.e.

$$\tau_c(\boldsymbol{k}) \equiv 2\int_0^\infty \phi_k^W(\tau)\,\mathrm{d}\tau. \tag{3.3}$$

The above approximations are in the same spirit as those made in Part 1, except that we are now concerned with the exact effect of the wave–wave interactions on the probability density.

3.1. *The steady-state probability density*

The first question of interest is whether the Fokker–Planck equation specifies a physically reasonable steady state for the resonant test-field model. Such a state is established at late times when the energy flux into a spectral region from the wind field and from the wave–wave interactions is balanced by the energy flux out of the region by wave–wave interactions and dissipation. The steady-state probability density $P_{ss}(\boldsymbol{a}) = \lim_{t\to\infty} P(\boldsymbol{a}, t|\boldsymbol{a}_0)$ describing this situation is independent of the initial configuration of the wave field and is independent of time, i.e.

$$\frac{\partial P_{ss}(\boldsymbol{a})}{\partial t} = \lim_{t\to\infty} \frac{\partial P(\boldsymbol{a}, t|\boldsymbol{a}_0)}{\partial t} = 0. \tag{3.4}$$

The steady-state solution to the Fokker–Planck equation (3.1) then satisfies the equation

$$\sum_k \left\{\frac{\partial}{\partial a_k}\left[\left(\frac{\lambda(\boldsymbol{k}) + \mathrm{i}\omega_k}{\omega_k}\right)\frac{\partial H_R}{\partial a_k^*}\right] + \text{c.c.} + 2D_k \frac{\partial^2}{\partial a_k\,\partial a_k^*}\right\} P_{ss}(\boldsymbol{a}) = 0. \tag{3.5}$$

An exact solution to (3.5) of the form

$$P_{ss}(\boldsymbol{a}) = Z^{-1} \exp\left\{-\sum_k \beta_k Q_k(\boldsymbol{a})\right\} \tag{3.6}$$

exists, where Z is the partition function, $\{\beta_k\}$ is a set of as-yet unknown parameters and $Q_k(\boldsymbol{a})$ is given by

$$Q_k(\boldsymbol{a}) \equiv \omega_k a_k a_k^* + \sum_{lmp} V_{mp}^{kl} a_k a_l a_m^* a_p^*. \tag{3.7}$$

The coupling coefficients V_{mp}^{kl} in (3.7) are given by the four-wave interaction strengths in the Hamiltonian (2.5). In fact $Q_k(\boldsymbol{a})$ is analogous to a single 'particle' energy including the energy due to interactions and is such that the total energy of the resonant test field of waves is

$$H_R = \sum_k Q_k(\boldsymbol{a}). \tag{3.8}$$

The algebraic details showing that (3.6) is indeed a solution of (3.5) are given in West (1982*a*).

If each of the parameters β_k in the set were equal to the same $\boldsymbol{k}$-independent constant β, then the solution (3.6) would become

$$P_{ss}(\boldsymbol{a}) = Z^{-1}\,\mathrm{e}^{-\beta H_R}. \tag{3.9}$$

Equation (3.9) is a canonical distribution for the RTF, with β interpreted as the scalar temperature characterizing the ambient waves, i.e. the 'heat bath'. This situation is analogous to the classical statistical-mechanical description of a many-body system. However, the choice of a single temperature β is not consistent with the present dynamic equations. To verify this we observe that a condition on the distribution (3.6) being the solution to (3.5) is that the parameters satisfy the equality

$$\frac{1}{\beta_k} = \frac{\omega_k D_k}{\lambda_R(\boldsymbol{k})}, \tag{3.10}$$

where $\lambda_R(\boldsymbol{k})$ is the real part of $\lambda(\boldsymbol{k})$. In addition to (3.10) we know that, for an asymptotic steady state to be established, a balance between the energy supplied to a spectral interval of RTF waves by the fluctuations must be balanced by the energy being dissipated in that spectral interval. Thus by multiplying (3.5) on the left by $|a_k|^2$ and integrating over all of phase space, we obtain the fluctuation–dissipation relation

$$\frac{D_k}{\lambda_R(\boldsymbol{k})} = \langle |a_k|^2 \rangle_{ss} + \frac{1}{\omega_k} \sum_{lmp} V_{lm}^{kp} \langle a_k^* a_p^* a_l a_m \rangle_{ss}\, \delta_{k+p-l-m}, \tag{3.11}$$

which together with (3.10) yields

$$\frac{1}{\beta_k} = \langle Q_k(\boldsymbol{a}) \rangle_{ss} \tag{3.12}$$

as the $\boldsymbol{k}$-dependent temperature. Therefore, using the probability density (3.6) to evaluate the steady-state spectrum (3.12), we obtain the implicit relation for the β's:

$$\frac{1}{\beta_k} = Z^{-1} \int \mathrm{d}\Gamma(\boldsymbol{a}) \left[\omega_k |a_k|^2 + \sum_{lmp} V_{lm}^{kp} a_k^* a_p^* a_l a_m \right] \exp\left\{ -\sum_q \beta_q Q_q(\boldsymbol{a}) \right\}. \tag{3.13}$$

The determination of β_k from this self-consistency relation is a familiar problem from statistical mechanics. The solution to such problems usually require expanding the integral in (3.13) to obtain an infinite sum of linked-cluster diagrams; the determination of the energy spectrum for the RTF waves is no exception. An expression formally equivalent to (3.13) is

$$\frac{1}{\beta_k} = -\frac{\partial}{\partial \beta_k} \ln Z, \tag{3.15}$$

so that the n-wave partition function Z completely determines $\{\beta_k\}$.

3.2. *The steady-state spectral density*

Although we cannot integrate (3.12) in general, we can approximate the true value of β_k by restricting the integral to a weighted self-interaction of the test $\boldsymbol{k}$-wave. For this single wave the weight of the interaction is zero, therefore to account in part for the interactions that are being omitted we weight the diagonal interaction strength $V_k (\equiv V_{kk}^{kk})$ in the Hamiltonian (2.5) by an element of volume αk^2, where α is a constant, and assume V_k is radially symmetric. Then by introducing the polar coordinates (J_k, θ_k),

$$a_k = J_k^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\theta_k}, \tag{3.14}$$

and integrating the restricted form of (3.13) over all the RTF waves except $\boldsymbol{k}$, we obtain

$$\frac{1}{\beta_k} \approx -\frac{\partial}{\partial \beta_k} \ln Z_k, \tag{3.15}$$

where the single-mode partition function is

$$Z_k \equiv \int_0^\infty \mathrm{d}J_k \exp\{-\beta_k [\omega_k J_k + \alpha k^2 V_k J_k^2]\}. \tag{3.16}$$

To evaluate (3.15) we expand the interaction terms in (3.16) to obtain

$$Z_k = \sum_{m=0}^{\infty} \frac{(-\beta_k \alpha k^2 V_k)^m}{m!} \int_0^{\infty} J_k^{2m} \mathrm{e}^{-\beta_k \omega_k J_k} \mathrm{d}J_k \tag{3.17}$$

$$= \sum_{m=0}^{\infty} \frac{(-\beta_k \alpha k^2 V_k)^m}{m!} \frac{\Gamma(2m+1)}{(\beta_k \omega_k)^{2m+1}}. \tag{3.18}$$

Substituting (3.18) into (3.15) and taking the indicated derivative, one obtains from the $m = 0, 1, 2$ terms the relation

$$\frac{1}{\beta_k} \approx \frac{\omega_k^2}{\alpha k^2 V_k} \frac{\Gamma(3)}{\Gamma(5)}, \tag{3.19}$$

and from (3.13) the spectral density

$$\langle J_k \rangle_{\mathrm{ss}} \equiv \langle |a_k|^2 \rangle_{\mathrm{ss}} \approx \frac{\Gamma(3)\, \omega_k}{\alpha k^2 V_k \Gamma(5)}. \tag{3.20}$$

For deep-water gravity waves the interaction strength is $V_k \sim k^3$, so that

$$\langle J_k \rangle_{\mathrm{ss}} \sim \frac{\omega_k}{k^5}, \tag{3.21}$$

and the energy-spectral density is given by

$$\Psi_{\mathrm{ss}}(\boldsymbol{k}) = \frac{\langle |a_k|^2 \rangle_{\mathrm{ss}}}{\omega_k / k} \sim k^{-4} \quad \text{gravity waves.} \tag{3.22}$$

The spectrum (3.22) has also been obtained by Phillips (1977), using a scaling argument, and by West (1982*a*).

3.3. *Approach to the steady state*

The determination of the steady-state energy-spectral density (3.22) is quite encouraging as it indicates that the asymptotic properties of the RTF waves are reasonable. The question remains as to how the field approaches this state in time. To answer this question we consider the transport equation for the non-equilibrium spectral density. One of the more recent discussions of this equation has been given by Longuet-Higgins (1976) for a narrow-spectrum nonlinear wave. The transport equation he obtains is based on a random-phase argument, and in the limit where the self-interaction strength becomes a constant it reduces to the cubic integro-differential equation obtained by Hasselmann (1962, 1963). This transport equation has the limitation that its steady-state solution is unphysical, i.e. the energy of the steady state becomes either negative or infinite in the continuum limit. To avoid this unphysical result we do not rely on the random-phase arguments of Longuet-Higgins, nor on the quasi-Gaussian assumption of Hasselmann, but instead we construct the transport equations directly from the Fokker–Planck equation (3.1).

We introduce the average quantity

$$N_k(t) \equiv \langle J_k ; t \rangle, \tag{3.23}$$

where the average in (3.23) is taken with respect to the solution to the Fokker–Planck equation (3.1). The transport equation for this quantity is then determined using

$$\frac{\partial N_k(t)}{\partial t} = \int a_k a_k^* \frac{\partial P}{\partial t} (\boldsymbol{a}, t | \boldsymbol{a}_0)\, \mathrm{d}\Gamma(\boldsymbol{a}). \tag{3.24}$$

Introducing (3.1) into (3.24) and integrating by parts yields the *exact* transport equation

$$\frac{\partial}{\partial t} N_k(t) + 2\lambda_R(\boldsymbol{k}) N_k(t) = 2D_k - 2\,\mathrm{Re}\left\{\frac{\lambda(\boldsymbol{k}) + \mathrm{i}\omega_k}{\omega_k} \sum_{plm} V_{lm}^{kp}\, \delta_{k+p-l-m} \langle a_k^* a_p^* a_l a_m; t\rangle\right\}. \tag{3.25}$$

The fourth-order moment on the right-hand side of (3.25) must be approximated because the time-dependent probability density is not known. In particular, one might here apply the argument developed by Longuet-Higgins (1976). If this is done one obtains the cubic terms in Hasselmann's transport equation for the gravity-wave field. Unlike these earlier expressions the resulting transport equation (3.25) is irreversible owing to the explicit dependence on the decay parameter $\lambda_R(\boldsymbol{k})$ and the energy flux D_k. This mechanism accounts for the replacement of the unphysical steady-state spectrum obtained earlier by the physical result (3.22) obtained as the steady-state solution to (3.25) in this approximation.

4. Discussion and conclusions

We have used a mode-coupled representation of the nonlinear water-wave field to model the solution of a broadband spectrum of gravity waves. The reversible Hamiltonian form of the equations of motion are replaced by a Markovian resonant test-wave model in which the non-resonant wave interactions are assumed to be well represented by Gaussian flux of action, a dissipative current and a modification in the interaction strength among the RTF waves. A linear model of the air–sea coupling is also included. The dynamics of the RTF waves are discussed in terms of the evolution of the probability density in the phase space of the test-wave field. In the Markov limit the phase-space equation of evolution is determined to be the Fokker–Planck equation, which yields the non-Gaussian steady-state distribution

$$P_{ss}(\boldsymbol{a}) = Z^{-1} \exp\left\{-\sum_k \beta_k \left[\omega_k |a_k|^2 + \sum_{lmp} V_{lm}^{kp}\, a_k^* a_p^* a_l a_m\, \delta_{k+p-l-m}\right]\right\}. \tag{4.1}$$

This distribution determines the approximate energy-spectral density to be k^{-4}.

The relaxation of the RTF toward the statistical steady state (4.1) is a consequence of the irreversibility of the Fokker–Planck equation. In the spirit of Boltzmann we introduce an H-function

$$H \equiv -\int \mathrm{d}\Gamma(\boldsymbol{a})\, P(\boldsymbol{a}, t) \ln \frac{P(\boldsymbol{a}, t)}{P_{ss}(\boldsymbol{a})}, \tag{4.2}$$

where the dependence of $P(\boldsymbol{a}, t)$ on initial conditions has been suppressed and the function $H \rightarrow 0$ as $t \rightarrow \infty$ and the RTF approaches its statistical steady state. To determine the approach to the steady state we first rewrite the Fokker–Planck equation (3.1) as

$$\frac{\partial P(\boldsymbol{a}, t)}{\partial t} = \sum_k D_k \frac{\partial}{\partial a_k}\left[\frac{\partial P(\boldsymbol{a}, t)}{\partial a_k^*} - \frac{P(\boldsymbol{a}, t)}{P_{ss}(\boldsymbol{a})} \frac{\partial P_{ss}(\boldsymbol{a})}{\partial a_k^*}\right] + \text{c.c.}, \tag{4.3}$$

where we have used (4.1) and the expressions (3.10) and (3.6). The time derivative of the H-function (4.2) can then be written, using (4.3), as

$$\frac{\mathrm{d}}{\mathrm{d}t} H[P(\boldsymbol{a}, t)] = \int \mathrm{d}\Gamma(\boldsymbol{a}) \sum_k \left|\frac{\partial \sigma(\boldsymbol{a}, t)}{\partial a_k}\right|^2, \tag{4.4}$$

where

$$\sigma(\boldsymbol{a},t) \equiv \ln \frac{P(\boldsymbol{a},t)}{P_{ss}(\boldsymbol{a})}. \tag{4.5}$$

It is apparent from (4.4) that for finite times

$$\frac{\mathrm{d}H[P(\boldsymbol{a},t)]}{\mathrm{d}t} > 0 \tag{4.6}$$

and that

$$\lim_{t\to\infty} \frac{\mathrm{d}H}{\mathrm{d}t} = 0. \tag{4.7}$$

Thus the functional $H[P(\boldsymbol{a},t)]$ increases monotonically in time until it reaches its steady-state value of zero, after which it remains constant (see e.g. Grabert & Weidlich 1980).

Irreversibility enters into the description of a general physical system in the derivation of the Boltzmann equation from the Liouville equation through the introduction of a probability hypothesis. Pawula (1967) has proven that, if any even moment (greater than the second) of a process vanishes, then the Fokker–Planck equation is the only logically consistent differential representation of the linear Boltzmann equation. Thus there exists a relation between the Markov property of the RTF necessary to derive the Fokker–Planck representation and the Stosszahl-ansatz necessary to make the Boltzmann equation irreversible. We can gain some insight into this relation by means of the following phase-space argument.

The density of systems in the interval of phase space $(\boldsymbol{a}, \boldsymbol{a}+\mathrm{d}\boldsymbol{a})$ at time t is given by the phase-space distribution function $\rho_f(\boldsymbol{a},t)$. The time rate of change of $\rho_f(\boldsymbol{a},t)$ determines how the ensemble of systems redistributes itself in phase space as a function of time. This change in time is determined by the equation of evolution for the dynamic variable $\boldsymbol{A}(t)$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}(t) = \boldsymbol{T}[\boldsymbol{A}(t),\boldsymbol{f}(t)], \tag{4.8}$$

where we have written (2.16) in vector form and collected all the linear and nonlinear terms into the function $\boldsymbol{T}$ in (4.8). Note that the dependence of $\boldsymbol{T}$ on the rapidly varying force $\boldsymbol{f}(t)$ has been made explicit, and similarly we denote the solution to (4.8) for a particular realization of $\boldsymbol{f}(t)$ as $\boldsymbol{A}_f(t)$. The phase-space distribution function is then given by

$$\rho_f(\boldsymbol{a},t) \equiv \delta^{(n)}(\boldsymbol{a}-\boldsymbol{A}_f(t)) \equiv \prod_{j=1}^{n} \delta(a_{k_j} - A_{k_j,f}(t)), \tag{4.9}$$

indicating that the phase-space vector $\boldsymbol{a}$ has non-zero values only along the trajectory corresponding to the solution of (4.8) for a particular $\boldsymbol{f}(t)$. It has been shown by a number of investigators (see e.g. Lax 1966*b*; Lindenberg *et al.* 1983) that the phase-space density function satisfies the equation

$$\frac{\partial}{\partial t}\rho_f(\boldsymbol{a},t) + \frac{\partial}{\partial \boldsymbol{a}}\cdot\{\boldsymbol{T}[\boldsymbol{a},\boldsymbol{f}(t)]\,\rho_f(\boldsymbol{a},t)\} + \text{c.c.} = 0, \tag{4.10}$$

which has the *appearance* of a Liouville equation for $\rho_f(\boldsymbol{a},t)$.

The phase-space density function has a fine structure associated with the fluctuations $\boldsymbol{f}(t)$ and therefore it cannot be interpreted as a probability density even though it satisfies (4.10). The probability density for the RTF waves is obtained from $\rho_f(\boldsymbol{a},t)$ by averaging over an ensemble of realizations of $\boldsymbol{f}(t)$, i.e.

$$P(\boldsymbol{a},t|\boldsymbol{a}_0) \equiv \langle \rho_f(\boldsymbol{a},t) \rangle. \tag{4.11}$$

This averaging procedure coarse-grains the *reversible* phase-space density function and the *reversible* equation of motion (4.10), to yield the probability density (4.11) and the Fokker–Planck equation (3.1). How one constructs (3.1) from (4.10) by this averaging procedure is fairly standard and may be found in a number of places (see e.g. Lindenberg *et al.* 1983).

The above discussion does not pretend to be complete. In fact, the relation between reversible and irreversible behaviour in physical systems is still one of the great unsolved problems in statistical physics. My intention here has been to indicate the strategy followed in the development of the phenomenological RTF model and to suggest how one might go about providing a more fundamental justification. Also the fact that one is able to find a physically reasonable steady-state spectrum and a transport equation which describes the relaxation of the wave field to this steady state suggests that further development of the RTF model is justified.

As a final point we reiterated the observation made in §1 that one cannot maintain both the Gaussian and nonlinear properties of the wave field if it is to be described by a Markov random process. So far we have relinquished Gaussianity in favour of nonlinearity and have obtained the nonlinear transport equation (3.25) to describe the evolution of the RTF. One could also adopt the philosophy used in Part 1 and include an average nonlinear interaction in the description so as to obtain a Gaussian wave field with effective parameters. This approach is more useful calculationally and results in a transport equation for the correlation matrix $\boldsymbol{Q}(t) = \langle \boldsymbol{a}\boldsymbol{a}^{\dagger}; t\rangle$ of the form

$$\frac{\partial \boldsymbol{Q}(t)}{\partial t} + \boldsymbol{M}\boldsymbol{Q}(t) + \boldsymbol{Q}(t)\,\boldsymbol{M}^{\dagger} = 2\boldsymbol{D}, \tag{4.12}$$

where the matrix $\boldsymbol{M}$ is determined by applying the method of statistical linearization to the nonlinear-mode rate equations. The two-point probability density is a complex multivariate Gaussian in this approximation with a time-dependent mean, i.e. $\langle \boldsymbol{a}\rangle = \mathrm{e}^{-\boldsymbol{M}t}\,\boldsymbol{a}_0$, and time-dependent correlations determined by (4.12). The transport equation corresponding to (3.25) is determined from the diagonal elements of $\boldsymbol{Q}(t)$ to be

$$\frac{\partial}{\partial t} N_{\boldsymbol{k}}(t) + 2\gamma_{\mathrm{R}}(\boldsymbol{k}, t)\, N_{\boldsymbol{k}}(t) = 2D_{\boldsymbol{k}}, \tag{4.13}$$

where the 'dissipation rate' $\gamma_{\mathrm{R}}(\boldsymbol{k}, t)$ is now time-dependent. Further details on this model will be published elsewhere. This model is mentioned here in order to emphasize that a self-consistent description of the gravity-wave field can be developed which is based on a Markov random field with Gaussian statistics, but such a description has a transport equation of the form (4.13) and not (3.25).

I would like to thank K. Watson and the other members of the Center for Studies of Nonlinear Dynamics for providing the daily stimulation to complete this work. I also thank the Office of Naval Research for the financial support of this work.

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